Lecture 11

14.1 - Functions of Several Variables

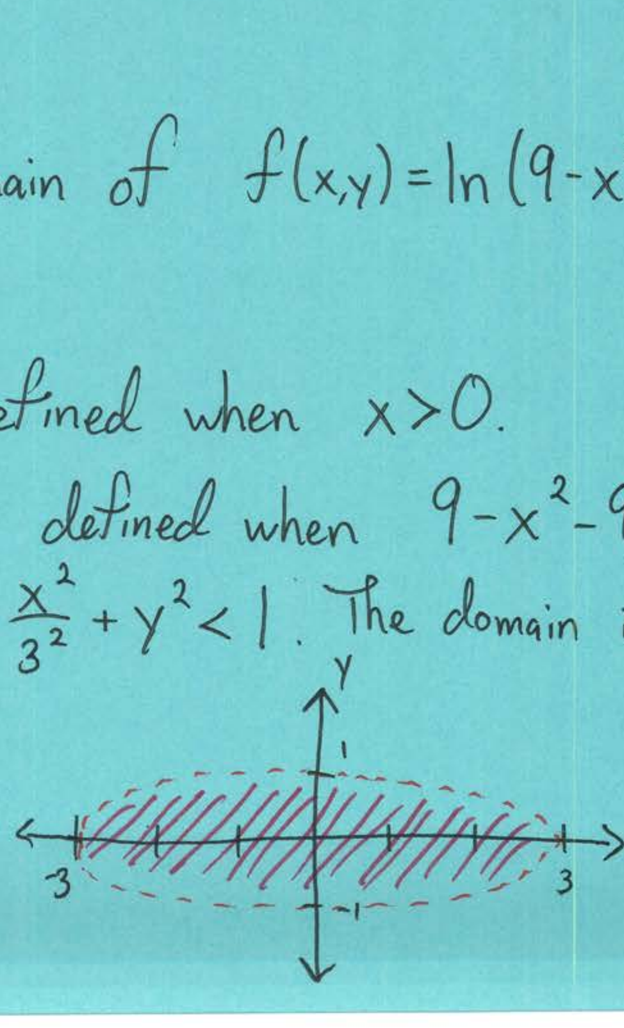
Functions of two variables

Given a function $f: D \longrightarrow \mathbb{R}$ defined on a set D of pairs $(x,y) \in \mathbb{R}^2$, called the domain of f, the range is the set of real numbers f takes on, i.e., $\{f(x,y) \mid (x,y) \in D\}$. If we set $z = f(x,y)$ we call x and y independent variables and z the dependent variable.

Ex: Find the domain of $f(x,y) = \ln(9 - x^2 - 9y^2)$ and sketch it.

Sol: $\ln(x)$ is defined when $x > 0$.
So $\ln(9 - x^2 - 9y^2)$ is defined when $9 - x^2 - 9y^2 > 0$
$\Leftrightarrow x^2 + 9y^2 < 9 \Leftrightarrow \frac{x^2}{3^2} + y^2 < 1$. The domain is: $D = \left\{(x,y) \,\middle|\, \frac{x^2}{3^2} + y^2 < 1\right\}$

So, the domain is graphically:



$\diamond$

<u>Ex 2</u>: Sketch the graph of $h(x,y) = 4x^2 + y^2$.

<u>Sol</u>: We want to graph all points (x,y,z) such that $h(x,y) = z$. So, we're graphing $z = 4x^2 + y^2$.

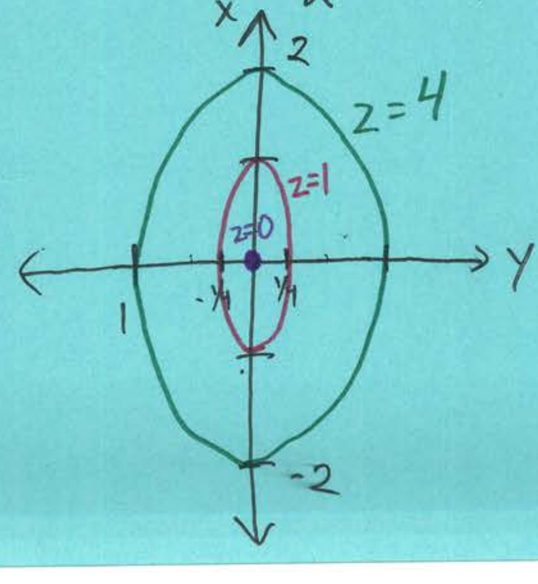
We see that $z \geq 0$ always, so to visualize this graph, we can examine what the function looks like at different z-values, called <u>level curves</u>

$\boxed{z=0}$: $4x^2 + y^2 = 0 \Rightarrow x = y = 0$. So, at $z=0$, we get a single point.

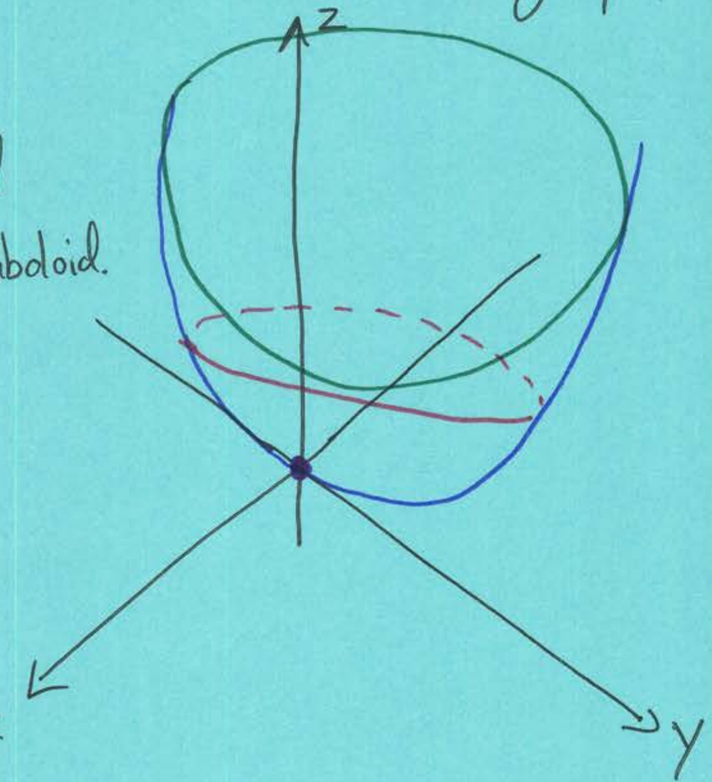
$\boxed{z=1}$: $4x^2 + y^2 = 1$ rewritten as $\dfrac{x^2}{(\frac{1}{2})^2} + y^2 = 1$

we see it as an ellipse with minor axis $\frac{1}{2}$ along x and major axis 1 along y.

$\boxed{z=4}$: $4x^2 + y^2 = 4 \Leftrightarrow x^2 + \dfrac{y^2}{2^2} = 1$, again an ellipse.



Lifting these up to the appropriate z-values we can start to see the graph of h:



This is called an elliptic paraboloid. ~~paraboloid~~

(Please excuse my bad drawing!) See mathematica code for something better.

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The 2D picture is called the <u>contour map</u> of h. The curves graphed, that is, the curves $h(x,y)=c$, are called <u>level curves</u> or <u>contours</u> of h. If h represents temperature, they are also called <u>isothermals</u>.

Functions of Three or More Variables

A function of three variables takes in triples (x,y,z) is a set $D\subset\mathbb{R}^3$, again called its domain, and outputs a real number $f(x,y,z)$

It is impossible to graph functions of three variables since it is 4-dimensional! We can, however, still study what $f(x,y,z)=c$ looks like for various values of c. This time, the sets $\{(x,y,z) \mid f(x,y,z)=c\}$ are called <u>level surfaces</u> since theyre two dimensional.

<u>Ex</u>: Find the domain of $f(x,y,z)=\sqrt{1-x^2-y^2-z^2}$.

<u>Sol</u>: We know $1-x^2-y^2-z^2 \geq 0$, which is the same as

$$x^2+y^2+z^2 \leq 1.$$

So, the domain is the solid sphere of radius 1.

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<u>Ex</u>: Find some level surfaces of $k(x,y,z)=x^2+3y^2+5z^2$.

<u>Sol</u>: Level surfaces of k look like

$$x^2+3y^2+5z^2=c^2=d$$

(Here, since $k \geq 0$ always, there is no harm in using c^2 for the constant since $c^2 \geq 0$.)

$$\Updownarrow$$

$$\frac{x^2}{c^2}+\frac{y^2}{\left(\frac{c}{\sqrt{3}}\right)^2}+\frac{z^2}{\left(\frac{c}{\sqrt{5}}\right)^2}=1$$

: This is an ellipsoid with "radius" c in the x-direction

$\frac{c}{\sqrt{3}}$ in the y-direction

$\frac{c}{\sqrt{5}}$ in the z-direction

$d = c^2 = 0 \Rightarrow x = y = z = 0$, a point.

$$d = c^2 = 1 \Rightarrow x^2 + \frac{y^2}{\left(\frac{1}{\sqrt{3}}\right)^2} + \frac{z^2}{\left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

$$d = 15 = c^2 \Rightarrow \frac{x^2}{(\sqrt{15})^2} + \frac{y^2}{(\sqrt{5})^2} + \frac{z^2}{(\sqrt{3})^2} = 1$$

$c = \sqrt{3}\sqrt{5}$
$= \sqrt{15}$

~~See mathematica code for graphs.~~

etc.

$\Diamond$

Just as before, we can just keep adding independent variables: $f(x_1, x_2, \cdots, x_n)$